DAΦNE TECHNICAL NOTE

INFN - LNF, Accelerator Division

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Note: **G-59**

Tune Shift in Beam-Beam Collisions with an Arbitrary Crossing Angle

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1. Introduction

In the previous Note [1] we have got analytical expressions for beam-beam tune shifts in collisions with a horizontal crossing angle. In this Note we generalize the analytical formulae for the case of an arbitrary crossing angle and check their validity by numerical simulations.

1. Beam-beam tune shift formulae

Let us consider two ultrarelativistic bunches colliding at an arbitrary angle, as shown in Fig. 1. The strong beam moves along z-axis of the right laboratory coordinate system. The coordinate system connected with the test particles of the weak beam is denoted with the index 'p' in Fig. 1. The coordinate transformations between the two systems are obtained, first, by a rotation of the strong bunch coordinate system by the angle ϕ around x-axis and, second, by a rotation of the resulting system x*-y*-z* around y*-axis by the angle θ.

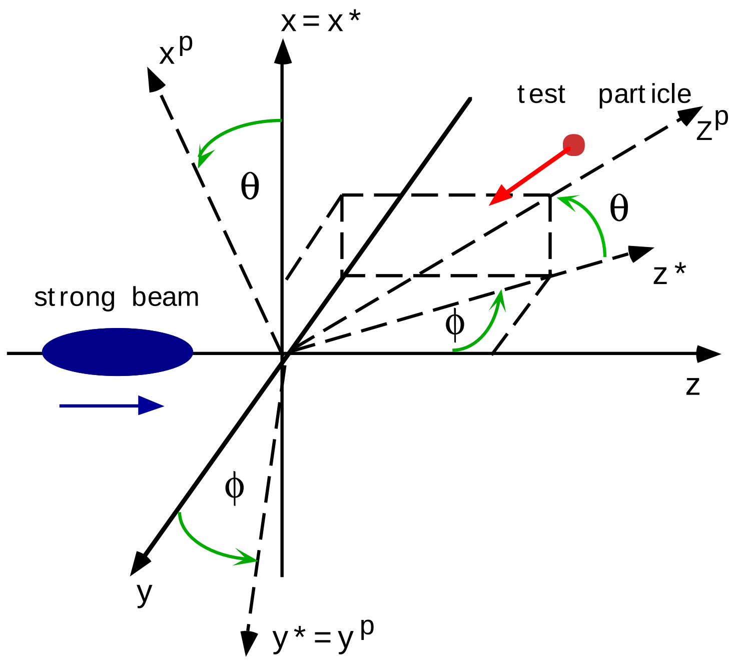

Figure 1: Scheme of beam-beam collision under a crossing angle.

The coordinate transformations from one system to the other are as follows:

$$
\begin{aligned}
&x = x^p \cos(\theta) + z^p \sin(\theta) \\
&y = y^p \cos(\phi) - \left(z^p \cos(\theta) - x^p \sin(\theta)\right)\sin(\phi) \\
&z = y^p \sin(\phi) + \left(z^p \cos(\theta) - x^p \sin(\theta)\right)\cos(\phi) \\
&and \\
&x^p = x\cos(\theta) - \left(z\cos(\phi) - y\sin(\phi)\right)\sin(\theta) \\
&y^p = y\cos(\phi) + z\sin(\phi) \\
&z^p = \left(z\cos(\phi) - y\sin(\phi)\right)\cos(\theta) + x\sin(\theta)
\end{aligned} \tag{1}
$$

In the laboratory system components of the electromagnetic field, created by a 3D Gaussian bunch (strong bunch) moving with a velocity ~c is given by [2]:

$$
E_x = \frac{eN\gamma}{2\pi^{3/2}\varepsilon_0} x \int_0^\infty dw \frac{\exp\left\{-\frac{x^2}{(2\sigma_x^2+w)} - \frac{y^2}{(2\sigma_y^2+w)} - \frac{\gamma^2(z-ct)^2}{(2\gamma^2\sigma_z^2+w)}\right\}}{(2\sigma_x^2+w)^{3/2}\sqrt{(2\sigma_y^2+w)(2\gamma^2\sigma_z^2+w)}}
$$

$$
E_y = \frac{eN\gamma}{2\pi^{3/2}\varepsilon_0} y \int_0^\infty dw \frac{\exp\left\{-\frac{x^2}{(2\sigma_x^2+w)} - \frac{y^2}{(2\sigma_y^2+w)} - \frac{\gamma^2(z-ct)^2}{(2\gamma^2\sigma_z^2+w)}\right\}}{(2\sigma_y^2+w)^{3/2}\sqrt{(2\sigma_x^2+w)(2\gamma^2\sigma_z^2+w)}} \tag{2}
$$

$$
B_x = -\frac{E_y}{c}
$$

$$
B_y = \frac{E_x}{c}
$$

Equations of motion of a test particle belonging to the weak beam in this system are:

$$
\begin{aligned}
&x(t) = -c\sin(\theta)t + x_0 && v_x = -c\sin(\theta) \\
&y(t) = c\cos(\theta)\sin(\phi)t + y_0 && v_y = c\cos(\theta)\sin(\phi) \\
&z(t) = -c\cos(\theta)\cos(\phi)t + z_0 && v_z = -c\cos(\theta)\cos(\phi)
\end{aligned} \tag{3}
$$

The Lorentz force acting on the test particle due to the electromagnetic fields produced by the strong beam:

$$\vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right) \quad with \quad \vec{v} \times \vec{B} = -v_z B_y \vec{i} + v_z B_x \vec{j} + \left(v_x B_y - v_y B_x\right)\vec{k} \tag{4}$$

has the following components:

$$\begin{aligned}
F_x &= e\left(E_x - v_z B_y\right) = e\left(E_x + c\cos(\theta)\cos(\phi)B_y\right) = eE_x\left(1 + \cos(\theta)\cos(\phi)\right) \\
F_y &= e\left(E_y + v_z B_x\right) = e\left(E_x - c\cos(\theta)B_x\right) = eE_y\left(1 + \cos(\theta)\cos(\phi)\right) \\
F_z &= e\left(v_x B_y - v_y B_x\right) = e\left(-c\sin(\theta)\frac{E_x}{c} + c\cos(\theta)\sin(\phi)\frac{E_y}{c}\right) = e\left(E_y\cos(\theta)\sin(\phi) - E_x\sin(\theta)\right)
\end{aligned} \tag{5}$$

The force projected onto the axes of the test particle coordinate system has:

$$\begin{aligned}
F_x^p &= F_x\cos(\theta) + F_y\sin(\phi)\sin(\theta) - F_z\cos(\phi)\sin(\theta) = eE_x\left(\cos(\phi) + \cos(\theta)\right) + eE_y\sin(\theta)\sin(\phi) \\
F_y^p &= F_y\cos(\phi) + F_z\sin(\phi) = eE_y\left(\cos(\phi) + \cos(\theta)\right) - eE_x\sin(\theta)\sin(\phi)
\end{aligned} \tag{6}$$

According to the tune shift definitions:

$$\begin{aligned}
\xi_{x^p} &= \Delta Q_{x^p} = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dz^p \beta_x \left.\frac{\partial F_x^p\left(x\left(x^p, y^p, z^p\right), y\left(x^p, y^p, z^p\right), z\left(x^p, y^p, z^p\right)\right)}{\partial x^p}\right|_{x^p = y^p = 0} \\
\xi_{y^p} &= \Delta Q_{y^p} = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dz^p \beta_y \left.\frac{\partial F_y^p\left(x\left(x^p, y^p, z^p\right), y\left(x^p, y^p, z^p\right), z\left(x^p, y^p, z^p\right)\right)}{\partial y^p}\right|_{x^p = y^p = 0}
\end{aligned} \tag{7}$$

Combining eqs. (1), (2) and (6), differentiating with respect to the transverse coordinate (eq. (7)) and integrating along z^p, one gets:

$$\xi_{xp} = \frac{r_e N \beta_x}{2\pi\gamma} \times \int_0^\infty dw \frac{\left\{ \begin{array}{c} (\cos\theta + \cos\phi)^2 (1 + \cos\theta\cos\phi)(2\sigma_y^2 + w) + \sin\phi^2 (1 + \cos\theta\cos\phi)(2\sigma_z^2 + \frac{w}{\gamma^2}) \\ + \sin\theta^2 \sin\phi^2 \left(1 - \cos\theta^2 \cos\phi^2\right)(2\sigma_x^2 + w) \end{array} \right\}}{\left((1 + \cos\theta\cos\phi)^2 (2\sigma_x^2 + w)(2\sigma_y^2 + w) + \cos\theta^2 \sin\phi^2 (2\sigma_x^2 + w)(2\sigma_z^2 + \frac{w}{\gamma^2}) + \sin\theta^2 (2\sigma_y^2 + w)(2\sigma_z^2 + \frac{w}{\gamma^2}) \right)^{3/2}} \quad (8)$$

$$\xi_{yp} = \frac{r_e N \beta_y}{2\pi\gamma} \times \int_0^\infty dw \frac{\left\{ \begin{array}{c} (\cos\theta + \cos\phi)^2 (1 + \cos\theta\cos\phi)(2\sigma_x^2 + w) + \sin\phi^2 \sin\theta^2 (1 + \cos\theta\cos\phi)(2\sigma_y^2 + w) \\ + \sin\theta^2 \cos\phi \left(\cos\phi + \cos\theta - \cos\theta \sin\phi^2\right)(2\sigma_z^2 + \frac{w}{\gamma^2}) \end{array} \right\}}{\left((1 + \cos\theta\cos\phi)^2 (2\sigma_x^2 + w)(2\sigma_y^2 + w) + \cos\theta^2 \sin\phi^2 (2\sigma_x^2 + w)(2\sigma_z^2 + \frac{w}{\gamma^2}) + \sin\theta^2 (2\sigma_y^2 + w)(2\sigma_z^2 + \frac{w}{\gamma^2}) \right)^{3/2}}$$

Note, that for combinations *($\theta = 0$; $\phi = 0$)* and *($\theta = \pi$; $\phi = \pi$)* the above expressions are reduced to the well know formulae for the head-on collision. Besides, for arbitrary θ and $\phi = 0$ eqs. (8) reproduce the formulae (9) in [1] for the tune shifts with a horizontal crossing angle.

In case when $\gamma >> tg(\theta/2)$ we can neglect the term $w/(\gamma^2 ctg^2(\theta))$ and for the case of a horizontal crossing angle *($\phi = 0$)* we obtain:

$$\begin{aligned} \xi_{xp} &= \frac{r_e N}{2\pi\gamma} \frac{\beta_x}{\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} \left(\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} + \sigma_y \right)} \\ \xi_{yp} &= \frac{r_e N}{2\pi\gamma} \frac{\beta_y}{\sigma_y \left(\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} + \sigma_y \right)} \end{aligned} \quad (9)$$

Similarly, for the vertical crossing angle *($\theta = 0$)* we get:

$$\begin{aligned} \xi_{xp} &= \frac{r_e N}{2\pi\gamma} \frac{\beta_x}{\sigma_x \left(\sqrt{\left(\sigma_z^2 tg^2(\phi/2) + \sigma_y^2\right)} + \sigma_x \right)} \\ \xi_{yp} &= \frac{r_e N}{2\pi\gamma} \frac{\beta_y}{\sqrt{\left(\sigma_z^2 tg^2(\phi/2) + \sigma_y^2\right)} \left(\sqrt{\left(\sigma_z^2 tg^2(\phi/2) + \sigma_y^2\right)} + \sigma_x \right)} \end{aligned} \quad (10)$$

Considering the last expressions (8)-(9) and the luminosity formula given in [3]:

$$L = \frac{N^2}{4\pi\sigma_y\sqrt{\left(\sigma_z^2 tg^2(\theta/2)+\sigma_x^2\right)}} \tag{11}$$

we can see that both eqs. (9) and (11) can be obtained from similar formulae for the head-on collision by simply substituting:

$$\sigma_x \;-->\; \sqrt{\left(\sigma_z^2 tg^2(\theta/2)+\sigma_x^2\right)} \tag{12}$$

in case of collisions with a horizontal crossing angle and:

$$\sigma_y \;-->\; \sqrt{\left(\sigma_z^2 tg^2(\phi/2)+\sigma_y^2\right)} \tag{13}$$

for the collisions at a vertical angle.

1. Comparison with numerical simulations

In order to check the formulae (8) describing the general case of collisions with an arbitrary angle we use numerical simulations with LIFETRAC code [4]. The betatron tunes in the presence of beam-beam effects are calculated by tracking in the following way. First of all a test particle is tracked for one turn with the initial conditions:

$$X_i = \delta(i,j)\sigma_i q, \quad i = 1,2,..,6, \quad q << 1, \quad \delta(i,j) = \begin{cases} 0, & if \quad i \neq j \\ 1, & if \quad i = j \end{cases} \tag{14}$$

where X_i are the coordinates in the 6D phase space and σ_i are the respective rms sizes.

Doing this 6 times for j = 1,2,..,6 we obtain the 6X6 revolution matrix. Then, the matrix eigenvalues are calculated those give us the tunes. For these simulations we use a simple model of collider with linear transformations from IP to IP. In order to reproduce correctly the gaussian longitudinal distribution we divide a strong bunch in much more longitudinal slices than in ordinary beam-beam simulations. Besides, the longitudinal rms bunch length is taken to be much shorter than the transverse beta functions at IP in order to satisfy approximations made to obtain eqs. (8). In these conditions the following equation is valid:

$$\cos(\mu) = \cos(\mu_0) - 2\pi\xi\sin(\mu_0) \tag{15}$$

where μ_0 is the initial betatron tune (transverse or vertical, without beam-beam), and μ is the tune calculated by tracking. Thus, we can find the tune shift ξ.

Figure 2 shows the normalised vertical tune shifts calculated analytically and numerically for comparison, while Fig. 3 presents the results for the horizontal tune shifts. As it is seen, the agreement between the analytical formulae and the simulations is very much satisfactory.

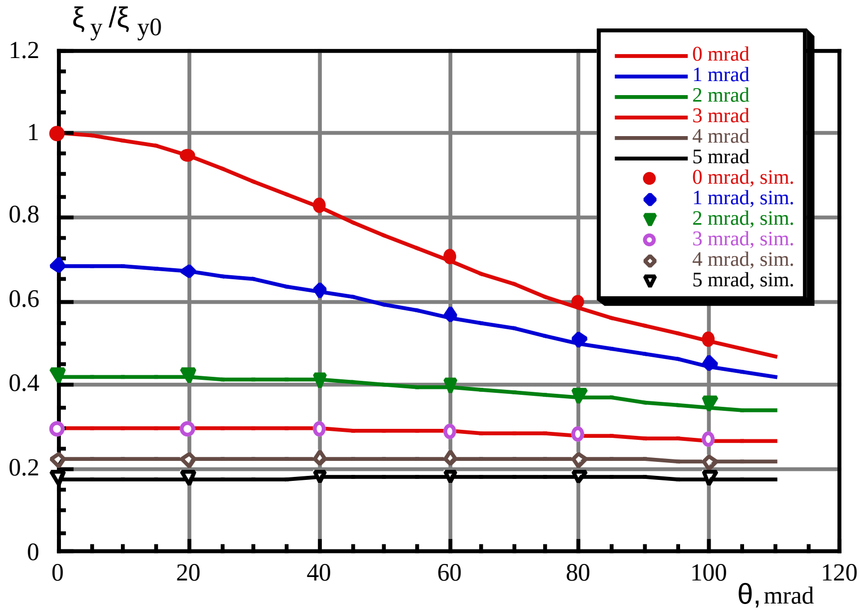

Figure 2: The vertical tune shift as a function of angle θ for ϕ = 0,1,2,3,4 and 5 mrad (normalised by the value of the vertical tune shift in head-on collisions). Solid lines – analytical results, dots – simulation results.

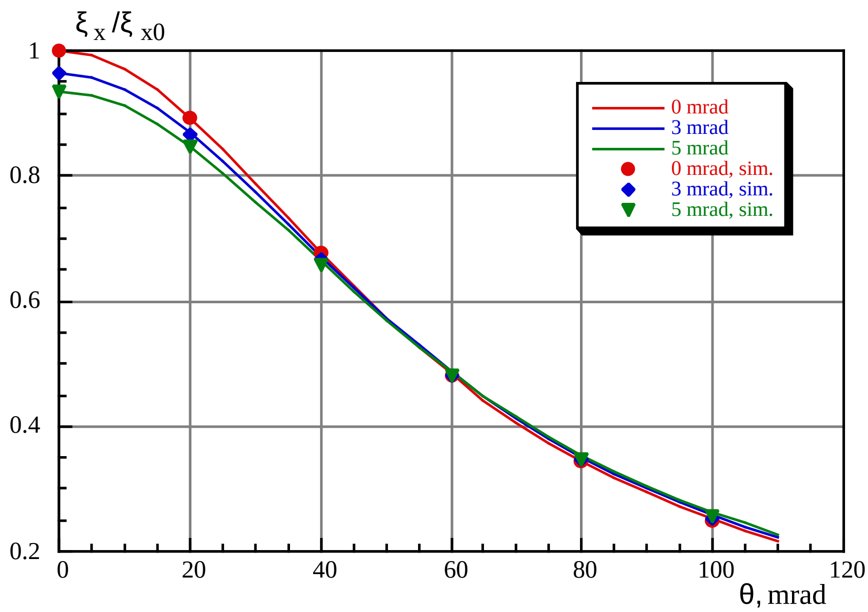

Figure 3: The horizontal tune shift as a function of angle θ for ϕ = 0,1,2,3,4 and 5 mrad (normalised by the value of the horizontal tune shift in head-on collisions). Solid lines – analytical results, dots – simulation results.

2. Conclusions

1. We have obtained the formulae for the beam-beam tune shifts in collisions with an arbitrary crossing angle. In particular, it has been shown that these formulae can be transformed from the similar formulae for head-on collisions by substituting the horizontal beam size by $\left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)^{1/2}$ in case of collisions with a horizontal crossing angle and the vertical beam size by $\left(\sigma_y^2+\sigma_z^2 tg^2(\phi/2)\right)^{1/2}$ if bunches collide at a vertical crossing angle.

2. Analysing the tune shift formulae, we see that for flat beams:

   a) the luminosity and the tune shifts are reduced with the horizontal crossing angle. However, since

   $$L \sim \left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)^{-1/2}; \quad \xi_x \sim \left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)^{-1}; \quad \xi_y \sim \left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)^{-1/2}$$

   the horizontal tune shift drops faster than the luminosity does.

   b) in collisions with the vertical crossing angle the horizontal tune shift practically does not depend on the vertical angle if $\sigma_x >> \left(\sigma_y^2+\sigma_z^2 tg^2(\phi/2)\right)^{1/2}$ while the vertical tune shift and the luminosity are reduces proportionally to $\left(\sigma_y^2+\sigma_z^2 tg^2(\phi/2)\right)^{-1/2}$.

3. A comparison of analytical tune shifts calculations with eqs. (8) and numerical simulations has shown a good agreement.

3. References

1. P. Raimondi and M. Zobov, DAΦNE Technical Note:G-58, Frascati, April 30, 2003.
2. A. A. Zholents, Preprint 91-18, Novosibirsk, 1991.
3. O. Napoly, Particle Accelerators, 1993, Vol. 40, pp. 181-203.
4. D. Shatilov, Particle Accelerators, 1996, Vol. 52, p.65.