DAΦNE TECHNICAL NOTE

INFN - LNF, Accelerator Division

Frascati, July 23, 1996

Note: **G-41**

IMPEDANCE OF A COAXIAL CAVITY COUPLED TO THE BEAM PIPE THROUGH A SMALL HOLE

S. De Santis, L. Palumbo

Abstract

In this paper we derive the impedance of a coaxial-line resonator coupled to the beam pipe through a small hole. The method used takes into account the scattered fields on the aperture to calculate its electric and magnetic dipole moments. The low frequency impedance shows a resistive contribution accounting for the cavity loss.

1. Introduction

The low frequency impedance of a hole on a beam pipe can be calculated by applying Bethe's diffraction theory, stating that the hole is equivalent to a combination of radiating electric and magnetic dipoles and that their moments are related to the amplitude of the incident field. This method, being independent from the structure geometry outside the beam pipe, yields an imaginary impedance only [1-3]. More recently, the real part of the impedance has been calculated taking into account the energy radiated by the hole through propagating fields [4-6]. In this paper we calculate the impedance when the hole radiates into a resonant structure (Fig. 1), as this geometry is more likely to represent properly many cases that are encountered in practice.

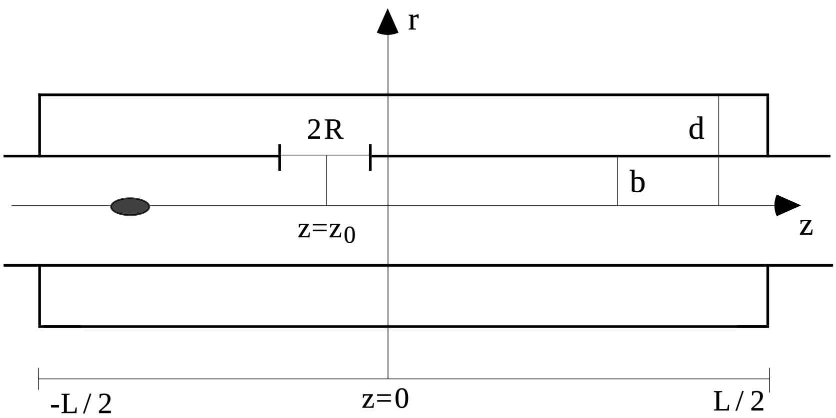

FIG. 1 - Coaxial resonator.

2. Monopole Longitudinal Impedance

It has been shown [1] that the longitudinal impedance of a hole in the wall of a round beam pipe can be expressed as a function of the magnetic and electric dipole moments, M_j and P_r, corresponding to a first order approximation of the scattered field. Limiting ourselves to frequencies below the pipe cutoff, we can write for a point charge q travelling along the pipe axis with velocity c:

$$Z_{//} = -j \frac{\omega Z_0}{2\pi b q}\left(\frac{1}{c} M_\varphi + P_r\right) \tag{1}$$

In general, the dipole moments are given by

$$\mathbf{P} = \varepsilon \breve{\alpha}_e \cdot (\mathbf{E}_0 + \mathbf{E}_{sp} - \mathbf{E}_{sc}),\ \mathbf{M} = \breve{\alpha}_m \cdot (\mathbf{H}_0 + \mathbf{H}_{sp} - \mathbf{H}_{sc}) \tag{2}$$

where $\breve{\alpha}_e$ and $\breve{\alpha}_m$ are the polarizability tensors for the aperture, $\mathbf{E}_0$ and $\mathbf{H}_0$ are the primary field radiated by the travelling particle (appendix A), and $\mathbf{E}_{sp}$, $\mathbf{H}_{sp}$, $\mathbf{E}_{sc}$, $\mathbf{H}_{sc}$ are the scattered fields in the pipe and in the cavity respectively. All the fields are evaluated at the aperture center ($r = b, \varphi = 0, z = z_0$).

The modified Bethe's diffraction theory [7] states that only the propagating modes contribute to the dipole moments in (2). Assuming that only the TEM_1 mode is resonating in the cavity, equations (2) become

$$P_r = \varepsilon \alpha_e (E_{0r} - E_{scr}),\ \ M_\varphi = \alpha_m (H_{0\varphi} - H_{sc\varphi}) \tag{3}$$

The scattered fields E_{scr} and $H_{sc\varphi}$ can be expressed through the cavity eigenfunctions e_1, h_1, and the coupling coefficients c_{e1}, c_{h1}:

$$E_{scr} = c_{e1} e_1,\ \ H_{sc\varphi} = c_{h1} h_1 \tag{4}$$

where [7]

$$\begin{aligned} c_{e1} &= \frac{-j\omega\mu k_1 h_1 M_\varphi + \omega^2 \mu \left(1 + \frac{1-j}{Q_1}\right) e_1 P_r}{k_1^2 - k_0^2 \left(1 + \frac{1-j}{Q_1}\right)} \\ c_{h1} &= \frac{j\omega k_1 e_1 P_r + k_0^2 h_1 M_\varphi}{k_1^2 - k_0^2 \left(1 + \frac{1-j}{Q_1}\right)} \end{aligned} \tag{5}$$

e_1 and h_1 are the TEM_1 normalized mode fields calculated on the aperture center, that is

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{\pi L \ln(d/b)}} \frac{\cos(k_1 z_0)}{b} \\ h_1 &= -\frac{1}{\sqrt{\pi L \ln(d/b)}} \frac{\sin(k_1 z_0)}{b} \end{aligned} \tag{6}$$

Substituting (4) in (3), we obtain the following linear system for the dipole moments

$$\begin{pmatrix} 1+\alpha_e \frac{k_0^2}{\tilde{k}} \tilde{q} e_1^2 & j\alpha_e \omega\mu\varepsilon \frac{k_1}{\tilde{k}} e_1 h_1 \\ -j\alpha_m \omega \frac{k_1}{\tilde{k}} e_1 h_1 & 1+\alpha_m \frac{k_0^2}{\tilde{k}} h_1^2 \end{pmatrix} \begin{pmatrix} P_r \\ M_\varphi \end{pmatrix} = \begin{pmatrix} \alpha_e \varepsilon E_{0r} \\ \alpha_m H_{0\varphi} \end{pmatrix} \tag{7}$$

where, for the sake of compactness, we have defined

$$\tilde{q} = 1 + \frac{1-j}{Q_1}, \quad \tilde{k} = k_1^2 - k_0^2 \tilde{q} \tag{8}$$

After a few calculations, the longitudinal impedance is found to be

$$Z_{//} = -jZ_K F(z_0, \omega, Q_1) \tag{9}$$

where

$$-jZ_K = -j\frac{k_0 Z_0 R^3}{6\pi^2 b^2} \tag{10}$$

is the impedance derived by Kurennoy in [1] for a round hole in a circular pipe, and

$$F(z_0,\omega,Q_1) = \\ = \frac{1 - 4\eta \frac{k_0^2}{\tilde{k}} \left[1 + \cos^2(k_1 z_0)(\tilde{q}-1)\right]}{1 - 2\eta \frac{k_0^2}{\tilde{k}} \left[\cos^2(k_1 z_0)\tilde{q} - 2\sin^2(k_1 z_0) - 4\eta\cos^2(k_1 z_0)\sin^2(k_1 z_0)\right]} \tag{11}$$

$$\text{with } \eta = \frac{(R/b)^2 (R/L)}{3\pi \ln(d/b)}$$

In Figs. 1-2 the real and imaginary parts of the longitudinal impedance are shown for three positions of the hole. It is worth nothing that, as the hole moves from the middle to the side of the cavity, the impedance increases since there is coupling through the magnetic field as well. The frequency shift of the curves can be explained in terms of Slater's theorem.

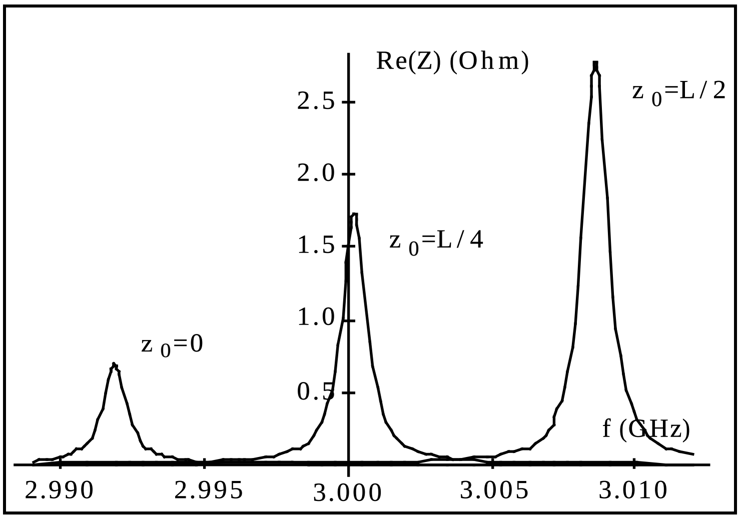

FIG. 2. Real part of the longitudinal impedance for three values of z_0 (L=50 mm, d=24 mm, b=20 mm, R=4 mm, Q_1=2900).

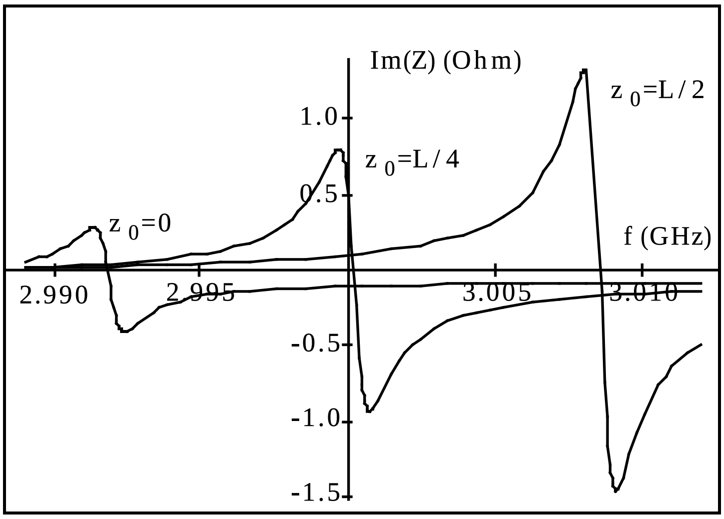

FIG. 3. Imaginary part of the longitudinal impedance for three values of z_0 (L=50 mm, d=24 mm, b=20 mm, R=4 mm, Q_1=2900).

A. Maximum shunt impedance

It is interesting to calculate the maximum value of the impedance as a function of the position z_0 of the hole. When z_0 = 0, that is the hole is at the cavity mid-length, it is easy to show that the real part of the longitudinal impedance is

$$Z_{RE} = \frac{2 Z_K \eta k_1^2 k_0^2 Q_1^{-1}}{\left[k_1^2 - k_0^2 (1 + 2\eta)(1 + Q_1^{-1})\right]^2 + \left[k_0^2 (1 + 2\eta) Q_1^{-1}\right]^2} \tag{12}$$

and that its maximum value

$$Z_{RE,\max} = \frac{2Z_K \eta(Q_1+1)}{1+2\eta} \approx 2\eta Q_1 Z_K \approx Z_0 \eta^2 Q_1 \ln(d/b) \tag{13}$$

is reached when

$$k_0 = \frac{k_1}{\sqrt{(1+2\eta)(1+Q_1^{-1})}} \tag{14}$$

The imaginary impedance is given by

$$Z_{IM} = -jZ_K \left\{ 1 - 2\eta k_0^2 \frac{k_1^2(1+Q_1^{-1}) - k_0^2(1+2\eta)(1+2Q_1^{-1})}{\left[k_1^2 - k_0^2(1+2\eta)(1+Q_1^{-1})\right]^2 + \left[k_0^2(1+2\eta)Q_1^{-1}\right]^2} \right\} \tag{15}$$

so that it is zero when (14) holds.

When the hole is no more at the cavity mid-length, we can see from (9) that for low-loss cavities and η<<1

$$\frac{Z_{RE,\max}(z_0)}{Z_{RE,\max}(z_0=0)} = 1 + 3\sin^2(k_1 z_0) \tag{16}$$

3. Dipole Longitudinal and Transverse Impedance

Proceeding in a similar manner, one can easily derive the transverse and the dipole longitudinal impedances by applying their standard definitions, provided the expressions of the dipole component of the incident field are used in the right hand side of system (7).

We obtain for a point charge with offset r_1,φ_1

$$Z_{//}^{n=1}(r,\varphi) = -j\frac{2k_0 Z_0}{3\pi^2}\frac{R^3}{b^4}F(z_0,\omega,Q_1) r r_1 \cos\varphi\cos\varphi_1 \tag{17}$$

and

$$\mathbf{Z}_\perp = -j\frac{2Z_0}{3\pi^2}\frac{R^3}{b^4}F(z_0,\omega,Q_1)\cos\varphi_1 \hat{\mathbf{r}} \tag{18}$$

Again, we find the same expressions found by Kurennoy, but for the factor $F(z_0,\omega,Q_1)$.

4. Conclusions

Applying Bethe's modified theory of diffraction to a hole radiating into a bounded space, we obtain that Kurennoy's impedance for a round hole is corrected by a complex coupling factor depending on the geometry and electromagnetic properties of the outer structure.

The correcting factor has been calculated for the case of a resonant coaxial structure, and the most relevant features of the low frequency coupling impedance have been investigated.

APPENDIX A

The fields produced by a point charge q travelling inside a perfectly conducting cylindrical pipe with velocity $c\hat{\mathbf{z}}$, can be expressed as a sum of multipole terms [8].

The low-frequency expression of the first (monopole) term on the pipe surface is

$$\begin{aligned} E_{0r}(r=b,\varphi=0) &= Z_0 \frac{q}{2\pi b} \\ H_{0\varphi}(r=b,\varphi=0) &= \frac{q}{2\pi b} \end{aligned} \tag{A1}$$

while the second (dipole) term is given by

$$\begin{aligned} E_{0r}^{n=1}(r=b,\varphi=0) &= Z_0 \frac{q}{2\pi^2 b^2} r_1 \cos\varphi_1 \\ H_{0\varphi}^{n=1}(r=b,\varphi=0) &= \frac{q}{2\pi^2 b^2} r_1 \cos\varphi_1 \end{aligned} \tag{A2}$$

APPENDIX B

The quality factor for a coaxial-line cavity of length L and radii b and d, resonating in the TEM_1 mode is

$$Q_1 = \frac{2L}{\delta\left(4 + \dfrac{L(1+d/b)}{d\ln(d/b)}\right)} \tag{B1}$$

The skin depth δ has the following expression

$$\delta = \sqrt{2}\,\frac{c}{\omega}\left[\sqrt{1+(\sigma/\omega\varepsilon)^2} - 1\right]^{-1/2} \tag{B2}$$

where σ is the conductivity of the cavity walls.

APPENDIX C

The resonant modes of a coaxial cavity can be obtained from the modes propagating into a coaxial line, with the additional condition of vanishing tangential electric field and normal magnetic field on the end plates (z = ±L/2).

The following modes are found:

TEM modes

$$\begin{aligned} \mathbf{e}_l &= C_l \frac{\cos(k_l z)}{r}\hat{\mathbf{r}} \\ \mathbf{h}_l &= -C_l \frac{\sin(k_l z)}{r}\hat{\varphi} \end{aligned} \tag{C1}$$

TE modes

$$\begin{aligned} \mathbf{e}_{n,m,l} &= C_{n,m,l}\Big(-\frac{n}{r}[J_n]\cos(n\varphi)\cos(k_l z)\hat{\mathbf{r}} + \\ &\qquad + k_{t(n,m)}[J'_n]\sin(n\varphi)\cos(k_l z)\hat{\varphi}\Big) \\ \mathbf{h}_{n,m,l} &= C_{n,m,l}\Bigg(-\frac{k_{t(n,m)}k_l}{\omega\mu}[J'_n]\sin(n\varphi)\sin(k_l z)\hat{\mathbf{r}} + \\ &\qquad - \frac{k_l}{\omega\mu}\frac{n}{r}[J_n]\cos(n\varphi)\sin(k_l z)\hat{\varphi} + \\ &\qquad + \frac{k^2_{t(n,m)}}{j\omega\mu}[J_n]\sin(n\varphi)\cos(k_l z)\hat{\mathbf{z}}\Bigg) \end{aligned} \tag{C2}$$

TM modes

$$\begin{aligned} \mathbf{e}_{n,m,l} &= C_{n,m,l}\Big(-\frac{k_{t(n,m)}}{\omega\varepsilon}[J'_n]\cos(n\varphi)\cos(k_l z)\hat{\mathbf{r}} + \\ &\qquad + \frac{1}{\omega\varepsilon}\frac{n}{r}[J_n]\sin(n\varphi)\cos(k_l z)\hat{\varphi} + \\ &\qquad + \frac{1}{j\omega\varepsilon}\frac{k^2_{t(n,m)}}{k_l}[J_n]\cos(n\varphi)\sin(k_l z)\hat{\mathbf{z}}\Big) \\ \mathbf{h}_{n,m,l} &= C_{n,m,l}\Big(-\frac{1}{k_l}\frac{n}{r}[J_n]\sin(n\varphi)\sin(k_l z)\hat{\mathbf{r}} + \\ &\qquad - \frac{k_{t(n,m)}}{k_l}[J'_n]\cos(n\varphi)\sin(k_l z)\hat{\varphi}\Big) \end{aligned} \tag{C3}$$

In the above expressions (C1-3) we have defined $k_l = l\pi/L$ and

$$
[J_n] = J_n(k_{t(n,m)}r) + \begin{cases} -\dfrac{J'_n(k_{t(n,m)}b)}{Y'_n(k_{t(n,m)}b)} Y_n(k_{t(n,m)}r) & \mathrm{TE}_{\mathrm{n,m,l}} \\ -\dfrac{J_n(k_{t(n,m)}b)}{Y_n(k_{t(n,m)}b)} Y_n(k_{t(n,m)}r) & \mathrm{TM}_{\mathrm{n,m,l}} \end{cases}
$$

$$
[J'_n] = J'_n(k_{t(n,m)}r) + \begin{cases} -\dfrac{J'_n(k_{t(n,m)}b)}{Y'_n(k_{t(n,m)}b)} Y'_n(k_{t(n,m)}r) & \mathrm{TE}_{\mathrm{n,m,l}} \\ -\dfrac{J_n(k_{t(n,m)}b)}{Y_n(k_{t(n,m)}b)} Y'_n(k_{t(n,m)}r) & \mathrm{TM}_{\mathrm{n,m,l}} \end{cases} \tag{C4}
$$

The k_t's in (C4) are 1/b times the zeros of $[J'_n]$ (TE modes) and of $[J_n]$ (TM modes), calculated for r = b.

The normalization factors C_l and $C_{n,m,l}$ are found from the condition

$$
\int_b^d \int_0^{2\pi} \int_{-L/2}^{+L/2} \left| \mathbf{e}_{n,m,l} \right|^2 r dr d\varphi dz = 1 \tag{C5}
$$

for the TEM_1 mode $C_1 = [\pi L \ln(d/b)]^{-1/2}$.

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