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ANALYTICAL FORMULAE FOR MULTIPOLE POTENTIALS

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INTRODUCTION

Magnetic multipole elements play an important role in particle accelerators, and their potentials are the subject of this paper. The main purpose is to obtain accurated values of the magnetic field (satisfying Maxwell eq. exactly), which can be used in theoretical computations. For real dipoles and quadrupoles, made of iron and coils, this is almost impossible because of the limited accuracy in magnetic design codes, construction errors, magnetic measurements precision and, last but not least, computing time. As a consequence I decided to take into account only magnetic potentials created by currents in vacuum.

In the literature already exist papers which, in a sense, follow this point of view. A first one is by K. Halbach[1] published in 1980, which studies multipoles realized with REC (Rare Earth Cobalt) blocks. Another paper concerning wigglers made of REC by R.P. Walker [2] clearly inspired by Ref. [1] was published in 1987. The link between these two papers and the present one is given by two assumptions, explicitly mentioned in [1] :

1) The permeability of the material is unity, so that it may be treated like vacuum with an imprinted current density. This means that the field produced by different pieces superpose linearly.

2) All blocks are homogeneously magnetized, so that they may be completely represented by current sheets at their surfaces.

In [1] and [2] these two assumptions provide a good approximation to the properties of the REC blocks. To the contrary, we assume here that the currents in vacuum represent reality.

We started with numerical calculations, but in the following it was discovered that a given type of magnetic elements was mathematically manageable. The conceptual path from the numerical to the analytical approach was very exciting. This paper will discuss only the analytical one.

All formulae presented refer to the magnetic potential only, which contains all the necessary information.

It will be shown that it is possible to find a mathematical treatment which holds at the same time for rectangular dipoles, quadrupoles, wigglers etc. Unfortunately the sector dipoles, very important for small accelerators, cannot be treated in this mathematical scheme.

1) THE MAGNETIC POTENTIAL OF MULTIPOLES

1.1) THE TWO AND THE THREE DIMENSIONAL POTENTIALS.

If the potential P depends only on two coordinates x,y (or r and α) only the Laplacian operator in two dimension is involved. We suppose to consider points in vacuum and the equation is:

$$\frac{\partial 2P}{\partial x2} + \frac{\partial 2P}{\partial y2} = 0 \qquad 1.1.1)$$

It is well known that particular solutions of this equation are the real or imaginary parts of :

$$Q_m(x,y) \propto (x+iy)^m = r^m \exp(i\ m\ \alpha) = r^m [\cos(m\ \alpha) + i \sin(m\ \alpha)] \qquad 1.1.2)$$

In the reality, the potential depends also on the third coordinate z. We try a more general solution by multiplying the two dimensions solution for a function $G_m(r,z)$ depending only on "r" and "z".

$$P_m(r,\alpha,z) = \frac{r^m \sin(m\ \alpha)}{m!}\ G_m(r,z) \qquad 1.1.3)$$

Obviously, if $G_m(r,z)$ tend to a constant $P_m(r,\alpha,z)$ tend to $P_m(r,\alpha)$. The potential $P_m(r,\alpha,z)$ will be called multipole potential of order m. In cylindrical coordinates $P_m(r,\alpha,z)$ and $G_m(r,z)$ must satisfy the equations:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P_m}{\partial r}\right) + \frac{\partial^2 P_m}{r^2 \partial \alpha^2} + \frac{\partial^2 P_m}{\partial z^2} = 0 \qquad 1.1.4)$$

and

$$\frac{\partial^2 G_m}{\partial r^2} + \frac{2m+1}{r}\frac{\partial G_m}{\partial r} + \frac{\partial^2 G_m}{\partial z^2} = 0 \qquad 1.1.5)$$

In the equation for G_m the first two terms are dimensionally divided by r^2, while the third one is not. This suggests an expansion in even powers of r for $G_m(r,z)$. Convergence for r=0 implies that we must omit negative powers of r. Hence we assume:

$$G_m(r,z) = \sum_{p=0}^{\infty} G_{m2p}(z)\ r^{2p} \qquad 1.1.6)$$

1.2) THE FORMAL SOLUTION OF THE THREE DIMENSIONAL POTENTIAL.

Substituting 1.1.6) in 1.1.5) gives:

$$\sum_{p=0}^{\infty} \left[4\,(m+p+1)(p+1)\,G_{m2p+2} + \frac{\partial^2 G_{m2p}}{\partial z^2}\right] r^{2p} = 0 \qquad 1.2.1)$$

The coefficients of individual power of r^{2p} must vanish separately :

$$G_{m2p+2} = -\frac{1}{4(m+p+1)(p+1)} \frac{\partial^{2p} G_{m2p}}{\partial z^{2p}} \qquad 1.2.2)$$

and setting p=1,2....we obtain :

$$G_{m2} = -\frac{1}{4(m+1)1} \frac{\partial^2 G_{m0}}{\partial z^2} \qquad 1.2.3)$$

$$G_{m4} = -\frac{1}{4(m+2)2} \frac{\partial^2 G_{m2}}{\partial z^2} = \frac{1}{4^2(m+2)(m+1)\,2!} \frac{\partial^4 G_{m0}}{\partial z^4} \qquad 1.2.4)$$

In general :

$$G_{m2p}(z) = (-1)^p \frac{m!}{4^p(m+p)!p!} \frac{\partial^{2p} G_{m0}}{\partial z^{2p}} \qquad 1.2.5)$$

and the total potential becomes:

$$P_m(r,\alpha,z) = \frac{r^m \sin(m\,\alpha)}{m!} \sum_{p=0}^{\infty} G_{m2p}(z)\, r^{2p} \qquad 1.2.6)$$

The trigonometric factor sin(mα) can be obviously replaced by cos(mα). In reality the derivatives of $G_{mo}(z)$ and the higher terms linked to them are particularly large at the ends of the magnets. For this reason the higher terms are called fringing field terms. It must be pointed out that, once m and $G_{mo}(z)$ fixed, the whole solution is determined. By taking m times the derivative of $P_m(r,\alpha,z)$ with respect to r and setting successively r =0, $G_{mo}(z)$ can be obtained trough the eq.

$$\sin(m\,\alpha) G_{mo}(z) = \left[\frac{\partial^m P_m(r,\alpha,z)}{\partial r^m}\right]_{r=0} \qquad 1.2.7)$$

2) CURRENT DISTRIBUTIONS

2.1) CURRENT DISTRIBUTIONS GENERATING PURE MULTIPOLE POTENTIALS

A real quadrupole, for instance made with iron, generates a potential which despite its name is not a pure quadrupole one. It contains higher order components which can be harmful to beam dynamics. It will be shown here that pure multipole potentials can be created with appropriate current distributions in vacuum.

Let us consider a surface S invariant to rotation around the z axis, and extending from $-Z_L$ to Z_L. Possible examples are the lateral surface of a cylinder or the surface of a piece of sphere, containing the cylinder and sharing two circumferences at the boundary. In general the surface S is completely defined through a function r(z).

Similarly to the earth surface we can imagine meridians on S, which are the intersections of S with planes passing trough the z axis, and parallels which are the intersection of S with planes perpendicular to the z axis (see Fig. 1). Let us suppose N coils s_k adjacent to each other and lying on S. Each one consists of two meridians and two pieces of end parallels. Let us indicate the position of each coil by its azimuthal average angle:

$$\theta_k = 2\pi (k-1/2)/N \qquad k=1,2,.....N \tag{2.1.1}$$

and assume that the current in the coil depends on the order m of the multipole and on θ_k trough the equation:

$$\mathbf{I}_s(k)=\mathbf{I}_s \sin(m\, \theta_k) \tag{2.1.2}$$

The decomposition of the current distribution in a sum of coils is useful for the calculation of the potential. The potential at a point P_0 given by a coil is [3] :

$$\Delta P(P_0)= (\mu_0\, \mathbf{I}/4\, \pi)\ \Delta\omega \tag{2.1.3}$$

where $\Delta\omega$ is the solid angle under which the point P_0 sees the coil and **I** the coil current. Let us consider N points P_q having all equal cylindrical coordinates "r" and "z", and the azimuthal angle θ_q, coincident with the average angles defined by eq. 2.1.1). Calling $\Delta\omega(\theta_k,\theta_q)$ the solid angle under which P_q sees the coil s_k we get for potential:

$$P_{m,q}= (\mu_0\, \mathbf{I}_s/4\, \pi) \sum_k \Delta\omega(\theta_k,\theta_q) \sin (m\, \theta_k) \tag{2.1.4}$$

As a consequence of the rotational symmetry, $\Delta\omega(\theta_k,\theta_q)$ depends only on the angle $(\theta_k-\theta_q)$ and has the same parity of $\cos(\theta_k-\theta_q)$. If we change index putting:

$$n=k-q \tag{2.1.5}$$

we obtain by 2.1.1):

$$\theta_k-\theta_q=2\pi\, n/N=\beta_n \qquad 2.1.6)$$

$$\theta_k=\theta_{n+q}= 2\pi\,(n+q-1/2)/N = \beta n+\theta q \qquad 2.1.7)$$

and eq. 2.14) becomes :

$$P_{m,q}=(\mu_0\, \mathbf{I}_s/4\,\pi)\sum_n \Delta\omega(\beta_n)\,[\sin(m\,\beta_n)\cos(m\theta_q)+\cos(m\,\beta_n)\sin(m\theta_q)] \qquad 2.1.4bis)$$

Because of the parity of $\Delta\omega(\beta_n)$, the term containing $\sin(m\beta_n)$ does not contribute and we get:

$$P_{m,q}= (\mu_0\, \mathbf{I}_s/4\,\pi)\sin(\theta_q)\sum_n\Delta\omega(\beta_n)\cos(m\,\beta_n) \qquad 2.1.8)$$

Let now N tend to infinity. We substitute θ_q by "α" and the sum by an integral. Eq. 2.1.8) becomes:

$$P_m(r,\alpha,z)=\frac{\mu_0\mathbf{I}_s}{4\,\pi}\,\sin(m\,\alpha)\int_0^{2\pi}\frac{\partial\omega}{\partial\theta}\cos(m\theta)d\theta \qquad 2.1.9)$$

By the comparison of Eqs. 1.2.6) and 2.1.9) we note that both have the factor $\sin(m\alpha)$. On the other hand the integral of eq. 2.1.9) does not depend on α and the potential can be written as:

$$P_m(r,\alpha,z)=\frac{r^m\sin(m\,\alpha)}{m!}\,G_m(r,z) \qquad 2.1.10)$$

which is the potential of a pure multipole. In principle we should show that it is possible to extract the factor r^m from the integral of eq. 2.1.9) without introducing negative powers inside the remaining function $G_m(r,z)$. That will be shown in the next paragraph for the case of a cylinder surface.

2.2) CURRENT DISTRIBUTIONS GENERATING ANALYTICAL POTENTIALS

It will be shown in this paragraph that, if the surface S coincides with the lateral boundary of a cylinder of radius R, the integral 2.1.9) can be expressed analytically. The distance between a point $P_0(r,\alpha,z)$, where we want to calculate the potential, and a point $P(R,\beta,Z)$ of the cylinder is :

$$s = [R^2 + r^2 - 2\,R\,r\cos(\beta - \alpha) + (Z - z)^2]^{1/2} \qquad 2.2.1.)$$

Let us call P_p the projection of P_0 onto the z axis ; the angle δ between P_pP_0 and PP_0 is defined by the eq. :

$$\cos(\delta) = \frac{R - r\cos(\beta-\alpha)}{s} \qquad 2.2.2)$$

The surface element on the cylinder is:

$$dS = R\,d\beta\,dZ \qquad 2.2.3)$$

and the solid angle seen from P:

$$d\omega = \frac{dS}{s^2}\cos(\delta) = \frac{R - r\cos(\beta-\alpha)}{s^3}\,R\,d\beta\,dZ \qquad 2.2.4)$$

We put $\theta = \beta - \alpha$ and define :

$$g(r,\theta) = \frac{R - r\cos(\theta)}{s^3} \qquad 2.2.5)$$

$$\frac{\partial\omega}{\partial\theta} = g(r,\theta)\,R\,dZ \qquad 2.2.6)$$

Substituting 2.2.5) and 2.2.6) into 2.1.9) we get :

$$P_m(r,\alpha,z) = \frac{\mu_0 \mathbf{I}_s R}{4\pi}\sin(m\alpha)\int_{-Z_L}^{Z_L} dZ \int_0^{2\pi} g(r,\theta)\cos(m\theta)\,d\theta \qquad 2.2.7)$$

Comparison between 2.2.7) and 1.2.7) gives the eq. for $G_{m0}(z)$:

$$G_{m0}(z) = \frac{\mu_0 \mathbf{I}_s R}{4\pi}\int_{-Z_{min}}^{Z_{max}} dZ \int_0^{2\pi} \cos(m\theta)\left(\frac{\partial^m g(r,\theta)}{\partial r^m}\right)_{r=0} d\theta \qquad 2.2.8)$$

Instead of using the definite integral specified by eq. 2.2.8) it is convenient to use the indefinite one $G_{m0}(Z,z)$:

$$G_{m0}(Z,z) = \frac{\mu_0 \mathbf{I}_s R}{4\pi}\int dZ \int_0^{2\pi} \cos(m\theta)\left(\frac{\partial^m g(r,\theta)}{\partial r^m}\right)_{r=0} d\theta \qquad 2.2.9)$$

With this choice the mathematical treatment is easier. Finally, to obtain $G_{m0}(z)$ it will be sufficient to write:

$$G_{m0}(z) = G_{m0}(Z_{max},z) - G_{m0}(Z_{min},z) \qquad 2.2.10)$$

The derivatives with respect to r and the integration on θ appearing in 2.2.9) are performed as shown in Appendix 1. As a consequence eq. 2.2.9) can be expressed as:

$$G_{m0}(Z,z) = \frac{\mu_0 \mathbf{I}_s (2m-1)!! R^m}{2^{m+1}} \; I_m(t) \qquad 2.2.11)$$

Where t, $I_m(t)$ and A(t) are:

$$t = Z - z \qquad 2.2.12)$$

$$I_m(t) = \int \left[\frac{(2m+1)R^2}{A^{2m+3}} - \frac{m}{A^{2m+1}}\right] dt \qquad 2.2.13)$$

$$A(t) = (R^2 + t^2)^{1/2} \qquad 2.2.14)$$

The indefinite integration in 2.2.13) is performed in Appendix 2. Defining:

$$f_k(t) = \left[\frac{t}{(R^2+t^2)^{1/2}}\right]^{2k+1} \qquad 2.2.15)$$

the result is :

$$I_m(t) = \frac{1}{R^{2m}} \sum_{k=0}^{m} (-1)^k \frac{m+k+1}{2k+1} \binom{m}{k} f_k(t) \qquad 2.2.16)$$

and 2.2.11) becomes :

$$G_{m0}(Z,z) = \mu_0 \mathbf{I}_s \frac{(2m-1)!!}{R^m 2^{m+1}} \left\{\sum_{k=0}^{m} (-1)^k \frac{m+k+1}{2k+1} \binom{m}{k} f_k(t)\right\} \qquad 2.2.17)$$

2.3) HIGHER ORDER TERMS

In order to achieve $G_{m2p}(Z,z)$ for $p > 0$ according to 1.2.5), it is convenient to write 2.2.17) in a more manageable way. To simplify the treatment let us define:

$$F_{m0}(t) = \frac{(2m-1)!!}{2^{m+1}} \sum_{k=0}^{m} (-1)^{k} \frac{m+k+1}{2k+1} \binom{m}{k} f_k(t) \qquad 2.3.1)$$

$G_{m0}(Z,z)$ becomes:

$$G_{m0}(Z,z) = \frac{\mu_0 \mathbf{I}_s}{R^m} F_{m,0}(t) \qquad 2.3.2)$$

and 1.2.5):

$$G_{m2p}(Z,z) = \frac{\mu_0 \mathbf{I}_s}{R^m} \frac{(-1)^p m!}{4^p (m+p)!p!} \frac{\partial^{2p} F_{m0}}{\partial z^{2p}} \qquad 2.3.3)$$

Inside 2.3.3) the most difficult task is to perform the second order derivatives of the f_k according to 2.3.1). Note that it is equivalent to take the second order derivative with respect to z or t. The result is (see Appendix 3) :

$$\frac{\partial^2 f_k}{\partial z^2} = \frac{(4k^2+2k)f_{k-1} - (12k^2+12k+3)f_k + (12k^2+18k+6)f_{k+1} - (4k^2+8k+3)f_{k+2}}{R^2} \qquad 2.3.4)$$

We make the following remarks :

a) The second order derivative of f_k is still a linear combination of f_k. As a consequence derivatives of every order have the same property.

b) From 2.3.4) it appears that the second order derivative of f_q introduces f_{q-1}. Whatever the initial value of "q" can be, by taking successive derivatives we reach f_0, but at this point the decrease of the index must stop because we have from 2.3.4) :

$$\frac{\partial^2 f_0(t)}{\partial t^2} = -3f_0(t) + 6f_1(t) - 3f_2(t) \qquad 2.3.5)$$

and f_{-1} is not generated. Therefore at every order of derivation the lowest function that can appear is always f_0.

c) It is easy to verify that the sum of the four coefficients of 2.3.4) vanishes for every k. From 2.2.15) it results that asymptotically, abs(t)»R, every f_k tends to 1 and as a consequence all the even derivatives asymptotically vanish. Also G_{mo} asymptotically vanishes but this happens only after the application of the integration limits on Z, which have opposite sign.

Properties a) and b) suggest to associate a vector $\mathbf{D}_{m,p}$ of components $D_{m,p,k}$ to every derivative. In particular one can put:

$$F_{m0}(t) = \sum_{k=0}^{+\infty} D_{m,o,k} f_k(t) \qquad 2.3.6)$$

and:

$$\frac{\partial^{2p} G_{m0}}{\partial z^{2p}} = F_{m,p}(t) = \sum_{k=0}^{+\infty} D_{m,p,k} f_k(t) \qquad 2.3.7)$$

Inside 2.3.6) e 2.3.7) the upper limit of the sums can be set as $+\infty$ because $D_{m,p,k}$ vanishes beyond a finite value of k. By taking the second derivative of 2.3.7) one obtains:

$$F_{m,p+1}(t) = \sum_{k=0}^{+\infty} D_{m,p,k} \frac{\partial^2 f_k(t)}{\partial t^2} \qquad 2.3.8)$$

and by applying 2.3.4)

$$F_{m,p+1}(t) = \sum_{k=0}^{+\infty} D_{m,p,k} \, * \qquad 2.3.9)$$

$$* \frac{(4k^2+2k) f_{k-1} - (12k^2+12k+3) f_k + (12\,k^2+18k+6) f_{k+1} - (4k^2+8k+3) f_{k+2}}{R^2}$$

The 4 terms in eq. 2.3.9) can be considered as the elements $M_{k-1,k}$, $M_{k,k}$, $M_{k,k+1}$, $M_{k,k+2}$ of the k^{th} column of a matrix **M**. By comparison one gets:

$$M_{k-1,\,k} = (4k^2+2k)$$

$$M_{k,\,k} = -(12k^2+12k+3)$$

$$M_{k+1,\,k} = (12k^2+18k+6) \qquad 2.3.10)$$

$$M_{k+2,\,k} = -\,(4k^2+8k+3)$$

The following table shows the first elements of M according to 2.3.10).

MATRIX M

k	M_{k0}	M_{k1}	M_{k2}	M_{k3}	M_{k4}	M_{k5}	M_{k6}	M_{k7}	M_{k8}	M_{k9}	M_{k10}	M_{k11}
0	-3	6	0	0	0	0	0	0	0	0	0	0
1	6	-27	20	0	0	0	0	0	0	0	0	0
2	-3	36	-75	42	0	0	0	0	0	0	0	0
3	0	-15	90	-147	72	0	0	0	0	0	0	0
4	0	0	-35	168	-243	110	0	0	0	0	0	0
5	0	0	0	-63	270	-363	156	0	0	0	0	0
6	0	0	0	0	-99	396	-507	210	0	0	0	0
7	0	0	0	0	0	-143	546	-675	272	0	0	0
8	0	0	0	0	0	0	-195	720	-867	342	0	0
9	0	0	0	0	0	0	0	-255	918	-1083	420	0
10	0	0	0	0	0	0	0	0	-323	1140	-1083	506

Clearly, because of its definition, M does not depend on the multipole order "m" and the derivation order "p". The first column -3, 6,-3 gives the coefficients of the second derivative of f_0, the second column 6,-27,36,-15 those of the second derivative of f_1 and so on. If one rewrites 2.3.9) as a linear combination of f_k's, namely :

$$F_{m,p+1}(t) = \sum_{k=0}^{+\infty} D_{m,p+1,k} \frac{f_k}{R^2} \qquad 2.3.11)$$

by using 2.3.9) and 2.3.10) eq. 2.3.11) can be written as:

$$\mathbf{D}_{m,p+1} = \frac{1}{R^2} \mathbf{M}\, \mathbf{D}_{m,p} \qquad 2.3.12)$$

and applying the eq. 2.3.12) p times the fundamental equation 2.3.7) becomes:

$$G_{m2p}(Z,z) = \frac{\mu_0 \mathbf{I}_s \quad (-1)^p m!}{R^{m+2p} \; 4^p (m+p)!p!} \sum_{k=0}^{+\infty} [\mathbf{M}^p \mathbf{D}_{m0}]_k f_k(t) \qquad 2.3.13)$$

Finally applying the eq. 2.2.10) :

$$G_{m2p}(z) = \frac{\mu_0 \mathbf{I}_s \quad (-1)^p m!}{R^{m+2p} \; 4^p (m+p)!p!} \sum_{k=0}^{+\infty} [M^p D_{m0}]_k \Big(f_k(t) \Big)_{t_{min}}^{t_{max}} \qquad 2.3.14)$$

where from the definition 2.2.12) of t obviously:

$$t_{max} = Z_{max} - z = Z_L - z$$
$$t_{min} = Z_{min} - z = -Z_L - z \qquad 2.3.15)$$

3) THE FUNCTIONS $G_{m,2p}$ FOR DIFFERENT MULTIPOLES

3.1) THE FUNCTIONS G_{10}, G_{12}, G_{14}, G_{16} OF THE DIPOLE

From 2.3.1), 2.3.6) and 2.3.14) we compute the coefficients of f_k of G_{12p} up to the sixth derivative (omitting for sake of simplicity the factor $\frac{\mu_0 \mathbf{I}_s}{R^{1+2p}}$).

DIPOLE

	G_{10}	G_{12}	G_{14}	G_{16}
f_0	.5	.375	.35156251048	.3417968
f_1	-.25	-1.21875	-2.5585938263	-4.315185579
f_2	0	1.3125	6.7968752026	20.046386868
f_3	0	-.46875	-8.5546877549	-47.535400745
f_4	0	0	5.1953126548	64.018555164
f_5	0	0	-1.2304687867	-49.757080449
f_6	0	0	0	20.866699374
f_7	0	0	0	-3.6657715117

From the table, and inserting the factor $\frac{\mu_0 \mathbf{I}_s}{R^{1+2p}}$, we obtain:

$$G_{10}(z) = \frac{\mu_0 \mathbf{I}_s}{R^1} [.5f_0(t) - .25f_1(t)] \Big|_{t_{min}}^{t_{max}}$$

$$G_{12}(z) = \frac{\mu_0 \mathbf{I}_s}{R^3} [.375f_0(t) -1.21875f_1(t) +1.3125f_2(t) -.46875f_3(t)] \Big|_{t_{min}}^{t_{max}}$$

$$G_{14}(z) = \frac{\mu_0 \mathbf{I}_s}{R^5} [.35156251048\, f_0(t) - 2.5585938263\, f_1(t) + \qquad 3.1.1)$$

$$+6.7968752026\, f_2(t)-8.5546877549\, f_3(t)+5.1953126548 f_4(t)-1.2304687867 f_5(t) \Big|_{t_{min}}^{t_{max}}$$

$$G_{16}(z)= \frac{\mu_0 \mathbf{I}_s}{R^7} [.3417968\, f_0(t)+ .... - 3.6657715117\, f_7(t)] \Big|_{t_{min}}^{t_{max}}$$

And for the dipole potential, by applying eq. 1.2.6) we obtain:

$$P_1(x,y,z) = y\, [G_{10}(z)+G_{12}(z)r^2 +G_{14}(z)r^4+G_{16}(z)r^6+ ....] \qquad 3.1.2)$$

3.2) THE FUNCTIONS G_{20}, G_{22}, G_{24}, G_{26} OF THE QUADRUPOLE

Similarly to the case of the dipole we obtain for the quadrupole omitting for sake of simplicity the factor $\frac{\mu_0 \mathbf{I}_s}{R^{2+2p}}$:

QUADRUPOLE

	G_{20}	G_{22}	G_{24}	G_{26}
f_0	1.125	.78125	.71777345889	.69213868863
f_1	-1.	-3.4375001024	-6.5625001956	-10.510254161
f_2	.375	5.6250001676	22.456055357	59.031739711
f_3	0	-4.0625001211	-38.554688649	-172.72705496
f_4	0	1.01937500326	35.786133879	297.56836658
f_5	0	0	-17.226563013	-314.53858184
f_6	0	0	3.3837891633	201.33545409
f_7	0	0	0	-71.849122834
f_8	0	0	0	10.997314719

From the table, and inserting the factor $\frac{\mu_0 \mathbf{I}_s}{R^{2+2p}}$ we get:

$$G_{20}(z) = \frac{\mu_0 \mathbf{I}_s}{R^2} \left[\frac{9}{8} f_0(t) - f_1(t) + \frac{3}{8} f_2(t)\right] \Big|_{t_{min}}^{t_{max}}$$

$$G_{22}(z)= \frac{\mu_0 \mathbf{I}_s}{R^4} [.78125\ f_0(t) - 3.4375001024\ f_1(t) + 5.6250001676\ f_2(t) -$$

$$- 4.0625001211\ f_3(t) + 1.01937500326\ f_4(t)] \Big|_{t_{min}}^{t_{max}}$$

3.2.1)

$$G_{24}(z) = \frac{\mu_0 \mathbf{I}_s}{R^6} [.71777345889\ f_0(t) - 6.5625001956\ f_1(t)+ 22.456055357\ f_2(t) -$$

$$- 38.554688649\ f_3(t) + 35.786133879\ f_4(t) - 17.226563013\ f_5(t) + 3.3837891633\ f_6(t)] \Big|_{t_{min}}^{t_{max}}$$

$$G_{26}(z) = \frac{\mu_0 \mathbf{I}_s}{R^8} [.69213868863\ f_0(t) -.... + 10.997314719\ f_8(t)] \Big|_{t_{min}}^{t_{max}}$$

And by applying 1.2.6) the potential is:

$$P_2(x,y,z) = xy\, [G_{20}(z) + G_{22}(z)r^2 + G_{24}(z)r^4 + G_{26}(z)r^6 + \ldots\ldots] \qquad 3.2.2)$$

3.3) THE POTENTIAL OF SOLENOIDAL CASES

The solenoidal case corresponding to m=0 should have been considered as the first one. However, this is not the most important case and furthermore it is not so well defined as a dipole or a quadrupole.

First of all let us remark that the factor before $G_{m0}(z)$ in eq. 1.2.6) should be $\cos(m\,\alpha)$, instead of $\sin(m\alpha)$, which is meaningless for m=0. Also for the dipole and quadrupole one could have chosen $\cos(m\alpha)$ instead of $\sin(m\alpha)$, and with this choice we would have obtained the formula which describes the skew multipoles.

If we use also in this case the scheme of N coils lying on the surface S of a cylinder, since m=0, the current running through adjacent coils is always the same. In this case the two currents of adjacent meridians compensate each other exactly. The resulting total current distribution is reduced to the two end coils with opposite current. This case is not very interesting. Actually the solenoidal cases of some interest are a simple coil and a solenoid extending from $-Z_L$ to Z_L.

Let us consider the coil first. The magnetic field B_z at the point z, created by a coil of radius R and current $\mathbf{I_S}$ is:

$$B_z = \frac{\mu_0 \mathbf{I_S}\, R^2}{2[R^2+z^2]^{3/2}} \qquad 3.3.1)$$

Integrating from $-\infty$ to z the potential $G_{00}(z)$ on the z axis comes out to be:

$$G_{00}(z) = \frac{\mu_0 \mathbf{I_S}}{2} \left[\frac{z}{[R^2+z^2]^{1/2}} - 1\right] \qquad 3.3.2)$$

the additive constant of 3.3.2) having no influence on the fields.

Using definition 2.2.14) and 2.2.15):

$$f_k(z) = \left(\frac{z}{A}\right)^{2k+1} \qquad 3.3.3)$$

we can write:

$$G_{S0}(z) = \frac{\mu_0 \mathbf{I_S}}{2}\; f_0(z) \qquad 3.3.4)$$

Defining as particular cases of 2.3.6) and 2.3.7):

$$D_{S00} = .5, \quad D_{S0k} = 0 \quad k=1,2,\ldots\ldots\infty \qquad 3.3.5)$$

$$\mathbf{D}_{Sp} = \frac{(-1)^p}{4^p[(p)!]^2}\; \mathbf{M}^p\, \mathbf{D}_{S0} \qquad 3.3.6)$$

we can write the table of coefficients D_{Spk} in G_{S2p}.

COIL

	G_{S0}	G_{S2}	G_{S4}	G_{S6}
f_0	.5	.375	.3515625	.34179687755
f_1	0.	-.75	-1.875	-3.4179687755
f_2	0	.375	3.515625	12.509765718
f_3	0	0	-2.8125	-22.695312669
f_4	0	0	.8203125	22.080078290
f_5	0	0	0	-11.074218833
f_6	0	0	0	2.2558593918

Inserting the factor $\frac{\mu_0 \mathbf{I}_s}{R^{2p}}$, we obtain from the table:

$$G_{S0}(z) = \mu_0 \mathbf{I}_s \, [.5 \, f_0(z) \,]$$

$$G_{S2}(z) = \frac{\mu_0 \mathbf{I}_s}{R^2} \, [.375 \, f_0(z) - .75 \, f_1(z) + .375 \, f_2(z)]$$

3.3.7)

$$G_{S4}(z) = \frac{\mu_0 \mathbf{I}_s}{R^4} \, [.3515625 \, f_0(z) - 1.875 \, f_1(z) + 3.515625 \, f_2(z) - 2.8125 \, f_3(z) + .8203125 \, f_4(t)]$$

$$G_{S6}(z) = \frac{\mu_0 \mathbf{I}_s}{R^6} \, [.34179687755 \, f_0(z) + \ldots\ldots + \ldots .2.2558593918 \, f_6(z)]$$

And applying eq. 1.2.6) :

$$P_S(x,y,z) = [G_{S0}(z) + G_{S2}(z) r^2 + G_{S4}(z) r^4 + G_{S6}(z) r^6 + \ldots\ldots \,] \quad 3.3.8)$$

3.4) THE POTENTIAL OF THE RECTANGULAR SOLENOID

A solenoid can be considered as a set of coils which cover uniformly the space between $-Z_L$ and Z_L. Replacing $\mathbf{I}_S$ from 3.3.2) with :

$$\mathbf{dI}_s = I_R \frac{dZ}{2Z_L} \tag{3.4.1)}$$

The contribution $dG_{S0}(z-Z)$ due to an infinitesimal coil of length dZ placed at Z is given by:

$$dG_{S0}(z) = dZ \frac{\mu_0 I_R}{4Z_L} \frac{z - Z}{[R^2+(z-Z)^2]^{1/2}} \tag{3.4.2)}$$

and $G_{R0}(Z,z)$ of the rectangular solenoid:

$$G_{R0}(Z,z) = -\frac{\mu_0 \mathbf{I}_R}{4Z_L} \int dZ \frac{(Z - z)}{[R^2 +(Z-z)^2]^{1/2}} \tag{3.4.3)}$$

To obtain $G_{R2}(Z,z)$, $G_{R4}(Z,z)$ etc., we consider eq. 1.2.5) in the case m=0:

$$G_{R2p}(Z,z) = (-1)^p \frac{1}{4^p((p)!)^2} \frac{\partial^{2p} G_{r0}}{\partial z^{2p}} \tag{3.4.4)}$$

Introducing the variable:

$$t = Z - z \tag{3.4.5)}$$

We need the even derivatives of :

$$G_{R0}(t) = -\frac{\mu_0 \mathbf{I}_R}{4Z_L} (R^2 +t^2)^{1/2} \tag{3.4.6)}$$

In analogy to multipoles with $m \geq 0$ we can introduce also in this case a matrix $\mathbf{M}_R$ similar to the matrix defined by 2.3.10). On the other hand, since the initial vector has only one non vanishing component and the matrix $\mathbf{M_R}$ is used only for the solenoid, it is easier to perform the derivatives directly.

Defining :

$$p_k(t) = \frac{1}{(R^2 +t^2)^{k-1/2}} \tag{3.4.7)}$$

we obtain easily:

$$\frac{\partial^2 p_k(t)}{\partial t^2} = (4k^2 - 2k)p_{k+1} + (1 - 4k^2)R^2 p_{k+2} \tag{3.4.8)}$$

from which :

$$\frac{\partial^2 p_0}{\partial t^2} = R^2 p_2$$

$$\frac{\partial^4 p_0}{\partial t^4} = 12 R^2 p_3 - 15 R^4 p_4 \qquad 3.4.9)$$

$$\frac{\partial^6 p_0}{\partial t^6} = 360 R^2 p_4 - 1260 R^4 p_5 + 945 R^6 p_6$$

Taking also in account that, according to 3.4.4) the second derivative has to be multiplied by (-1/4), the fourth by (1/64) and the sixth by (-1/2304)) the $G_{R2k}(z)$ are:

$$G_{R0}(z) = \frac{\mu_0 \mathbf{I}_R R^2}{4Z_L} \Big(p_0(t)\Big)_{Z_L-z}^{-Z_L-z} \qquad 3.4.10)$$

$$G_{R2}(z) = -\frac{\mu_0 \mathbf{I}_R R^2}{4Z_L} \Big(.25 p_2(t)\Big)_{Z_L-z}^{-Z_L-z} \qquad 3.4.11)$$

$$G_{R4}(z) = \frac{\mu_0 \mathbf{I}_R R^2}{4Z_L} \Big(.1875\ p_3(t) - .234375\ p_4(t)\Big)_{Z_L-z}^{-Z_L-z} \qquad 3.4.12)$$

$$G_{R6}(z) = -\frac{\mu_0 \mathbf{I}_R R^2}{4Z_L} \Big(.15625\ p_4(t) - .546875\ p_5(t) + .41015625\ p_6(t)\Big)_{Z_L-z}^{-Z_L-z} \qquad 3.4.13)$$

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APPENDIX 1. ORDER "m" DERIVATIVE OF g(r)

Let us consider the function:

$$g(r) = \frac{R\text{-}r\cos(\theta)}{[R^2+r^2-2\,R\,r\cos(\theta)+(Z-z)^2]^{3/2}} \qquad \text{A1.1)}$$

In order to make successive calculations easier we define :

$$s = [R^2 + r^2 - 2\,R\,r\cos(\theta)+(Z-z)^2]^{1/2} \qquad \text{A1.2)}$$

$$f(r) = R\text{-}r\cos(\theta) \qquad \text{A1.3)}$$

$$h(r) = r - R\cos(\theta) \qquad \text{A1.4)}$$

and we can write simply :

$$g(r) = \frac{f(r)}{s^3} \qquad \text{A1.5)}$$

Denoting by an apex the derivative with respect to r, we get:

$$f' = -\cos(\theta) \qquad \text{A1.6)}$$

$$h' = 1 \qquad \text{A1.7)}$$

$$s' = h\,s^{-1} \qquad \text{A1.8)}$$

From eq. A1.2).....A1.7) setting r=0 gives:

$$A=s(0) =[R^2 +(Z-z)^2]^{1/2} \qquad \text{A1.9)}$$

$$f(0) = R \qquad \text{A1.10)}$$

$$g(0) = \frac{R}{A^3} \qquad \text{A1.11)}$$

$$h(0) = -R\cos(\theta) \qquad \text{A1.12)}$$

Taking the "m" order derivative of g(r) defined in A1.1) we get terms whose dependence on θ is proportional to $\cos^n(\theta)$ with n running from 0 to m. If we look to the integration over θ of 2.2.9) we already have a term cos(mθ). The integrals to be performed are of the kind:

$$T(m,n) = \int_0^{2\pi} \cos(m\theta)\cos^n(\theta)\, d\theta \qquad \text{A1.13)}$$

writing:

$$\cos(\theta) = \frac{[\exp(i\theta) + \exp(-i\theta)]}{2} \qquad \text{A1.14)}$$

and taking the n^{th} power of A1.14) the sum of the first and the last terms gives $\frac{\cos(n\,\theta)}{2^{n-1}}$. The other terms contain a lower integer multiple of θ. So we obtain:

$$T(m,n) = \frac{1}{2^{m-1}}\int_0^{2\pi} \cos(m\theta)\cos(n\theta)\, d\theta \ + const \int_0^{2\pi} \cos(m\theta)\cos[(n-2)\theta]\, d\theta \qquad \text{A1.15)}$$

T(m,n) does not vanish only if n is equal to its maximum value m:

$$T(m,m) = \frac{1}{2^{m-1}}\int_0^{2\pi} \cos^2(m\theta)\, d\theta \ = \frac{\pi}{2^{m-1}} \qquad \text{A1.16)}$$

We can now face the problem of computing $\partial^m g(r),\partial r^m)$. As shown in A1.10), g(0) does not contain any factor cos(θ). A factor cos(θ), as shown in A1.5), A1.6) A1.7) e A1.11), is generated only if one takes the derivative of either the function s or the function f. As f'' vanishes the derivative must be like:

$$\frac{\partial^m g}{\partial r^m} = \frac{a_m f\, h^m}{s^{2m+3}} + \frac{b_m f'\, h^{m-1}}{s^{2m+1}} \qquad \text{A1.17)}$$

The coefficient a_m, being the product with alternating sign of the odd numbers starting from 1, comes out to be:

$$a_m = (-1)^m\,(2m+1)!! \qquad \text{A1.18)}$$

b_m can be derived by keeping the terms that are deduced from the derivation of f separated.

We obtain therefore:

$$\frac{\partial g}{\partial r} = -\frac{3\ f\ h}{s^5} + \frac{f'}{s^3}$$

$$\frac{\partial^2 g}{\partial r^2} = \left(\frac{15\ f\ h^2}{s^7} - \frac{3\ f'h}{s^5}\right) - \frac{3\ f'h}{s^5}\ A \qquad 1.19)$$

$$\frac{\partial^3 g}{\partial r^3} = \left(-\frac{105\ f\ h^3}{s^9} + \frac{15\ f'\ h^2}{s^7}\right) + \frac{15\ f'\ h^2}{s^7} + \frac{15\ f'\ h^2}{s^7}$$

Every derivation adds a new term equal to the ones already obtained. Clearly after m derivations we find:

$$b_m = m\ a_{m-1} = m\ (-1)^{m-1}\ (2m-1)!! \qquad A1.20)$$

And grouping the two terms:

$$\frac{\partial^m g}{\partial r^m} = (-1)^m\ h^{m-1}\left\{\frac{(2m+1)!!f\ h}{s^{2m+3}} - \frac{m(2m-1)!!f'}{s^{2m+1}}\right\} \qquad A1.21)$$

Substituting to the functions their value for r =0, and using A instead of s(o) one obtains :

$$\left(\frac{\partial^m g}{\partial r^m}\right)_{r=0} = \cos^m(\theta)\left\{\frac{(2m+1)!!\ R^{m+1}}{A^{2m+3}} - \frac{m(2m-1)!!R^{m-1}}{A^{2m+1}}\right\} \qquad A1.22)$$

If we multiply A1.22) by cos(mθ) and we take into account eq. A1.16), we obtain for the integral over θ of eq. 2.2.9):

$$\int_0^{2\pi} \cos(m\theta)\left(\frac{\partial^m g}{\partial r^m}\right)_{r=0} d\theta = \frac{\pi}{2^{m-1}}\left\{\frac{(2m+1)!!\ R^{m+1}}{A^{2m+3}} - \frac{m(2m-1)!!R^{m-1}}{A^{2m+1}}\right\} \qquad A1.23)$$

APPENDIX 2. INTEGRATION ON THE LONGITUDINAL COORDINATE

In eq. 2.2.12) appears the indefinite integral:

$$I_m(t) = \int \left\{ \frac{(2m+1)R^2}{A^{2m+3}} - \frac{m}{A^{2m+1}} \right\} dt \qquad A2.1)$$

where:

$$A(t) = (R^2 + t^2)^{1/2} \qquad A2.2)$$

The two integrals of A2.1) are of the kind:

$$F_m = \int dt / A^{2m+1} \qquad A2.3)$$

Defining :

$$f_m(t) = [t/A]^{2m+1} \qquad A2.4)$$

A2.3) can be written as [4]:

$$F_m = \frac{1}{R^{2m}} \sum_{k=0}^{m-1} \frac{(-1)^k f_k(t)}{2k+1} \binom{m-1}{k} \qquad A2.5)$$

Applying eq. A2.5) the total integral inside A2.1) becomes:

$$I_m(t) = \frac{1}{R^{2m}} \left\{ \left[\sum_{k=0}^{m-1} \left(\frac{2m+1}{2k+1} \binom{m}{k} - \frac{m}{2k+1} \binom{m-1}{k} \right) \right] ()^k f_k(t) + (-1)^m f_m(t) \right\} \qquad A2.6)$$

Each coefficient inside the sum, exploiting well known properties of binomial coefficients, can be written as:

$$\frac{2m+1}{2k+1} \binom{m}{k} - \frac{m}{2k+1} \binom{m-1}{k} = \frac{m+k+1}{2k+1} \binom{m}{k} \qquad A2.7)$$

If k=m the result of A2.7) is equal to 1. Therefore we can eliminate the last term $(-1)^m f_m(t)$ inside A2.6) provided we extend the sum up to m. The result is :

$$I_m(,t) = \frac{1}{R^{2m}} \sum_{k=0}^{m} (-1)^k \frac{m+k+1}{2k+1} \binom{m}{k} f_k(t) \qquad A2.8)$$

APPENDIX 3. CALCULATION OF $f_k(t)$ SECOND ORDER DERIVATIVES

Equation 2.2.15) defines $f_k(t)$ that, writing simply f_k, is:

$$f_k = \frac{t^{2k+1}}{A^{2k+1}} \qquad \text{A3.1)}$$

where:

$$A(t) = (R^2 + t^2)^{1/2} \qquad \text{A3.2)}$$

By using the partial result:

$$\frac{\partial A}{\partial t} = \frac{t}{A} \qquad \text{A3.3)}$$

$$\frac{\partial f_k}{\partial t} = (2k+1)\, R^2\, t^{2k}\, A^{-(2k+3)} (2k+1)\, R^2\, t^{-3}\, f_{k+1} \qquad \text{A3.4)}$$

and applying the first derivative again:

$$\frac{\partial^2 f_k}{\partial t^2} = \frac{2k+1}{R^2}\left[-3\frac{R^4}{t^4}\, f_{k+1} + (2k+3)\frac{R^6}{t^6}\, f_{k+2}\right] \qquad \text{A3.5)}$$

In order to eliminate $(R/t)^{2n}$, A3.2) suggests to replace R^2 with $(A^2 - t^2)$ obtaining:

$$(R/t)^4 = (A/t)^4 - 2(A/t)^2 + 1 \qquad \text{A3.6)}$$

$$(R/t)^6 = (A/t)^6 - 3(A/t)^4 + 3(A/t)^2 - 1 \qquad \text{A3.7)}$$

From definition A3.1) we have:

$$(A/t)^{2n} f_k = (A/t)^{2n}\, (t^{2k+1}/A^{2k+1}) = (t^{2k+1-2n}/A^{2k+1-2n}) = f_{k-n} \qquad \text{A3.8)}$$

Substituting eq. A3.8) inside A3.5) we obtains, after grouping similar terms together:

$$\frac{\partial^2 f_k}{\partial t^2} = \frac{(4k^2+2k)\, f_{k-1} - (12k^2+12k+3) f_k + (12k^2+18k+6)\, f_{k+1} - (4k^2+8k+3)\, f_{k+2}}{R^2} \qquad \text{A3.9)}$$